Homework 8: P and NP

CSE 30151 Spring 2022

Due: Tuesday, April 26 at 11:59pm

Instructions

- Use this document to create a PDF file containing your solutions. Do this either by (1) printing this document, writing your solutions on it, and scanning[1] your work into a PDF; or (2) writing your solutions on this PDF digitally. Either way, please ensure that your work is clearly legible.
- If you need extra blank pages, feel free to insert them as needed. The amount of blank space provided beneath a problem does not necessarily indicate the expected length of your solution.
- You have the option to submit your solutions all at once or in parts; late penalties will only be applied to problems that are late. Individual problems cannot be submitted for grading in this way more than once.
- If you plan to submit some parts of your assignment late, before the deadline, upload a single PDF containing the problems you have completed so far. Do not include solutions to problems you want graded later for late credit. After the deadline, if you want to submit additional problems, add them to your original PDF and upload it again.[2]
- Submit your PDF file in Canvas under Assignments > Homework 8: P and NP. You may re-submit your work any number of times before the due date.

[1] For tips on scanning your work using your mobile device, see `https://help.gradescope.com/article/0chl25eed3-student-scan-mobile-device`.

[2] For tips on concatenating your old and new PDFs together, see `https://help.gradescope.com/article/tp9kl4yx4q-student-troubleshooting-submissions`.

1. In the *traveling salesman problem*, you are a salesman and are given a list of cities and the distances between each pair of cities. A *tour* is a route that visits each city exactly once and returns to the city you started from. Your task is to find the shortest possible tour through the cities.

   (a) Consider a version of this problem called *LINETSP* where the "cities" are points that lie along a one-dimensional line. Let $S = \{x_1, x_2, \ldots, x_k\}$ be this set of points, where each $x_i \in \mathbb{N}$. When traveling between two cities, you are allowed to pass through other cities on the line without "visiting" them.

   The decision version of this problem is to determine, given a value $D \in \mathbb{N}$, whether there is a tour through the points in S where the total distance traveled is no more than D.

$$\begin{aligned} \textit{LINETSP} = \{\langle S, D\rangle \mid\ & S = \{x_1, \ldots, x_k\} \text{ and there is a tour of } S \\ & \text{whose total length is no more than } D\} \end{aligned}$$

   The points in the encoding of S are presented in no particular order.

   - Prove that *LINETSP* $\in$ P by providing a high-level description of a deterministic polynomial-time algorithm that decides it. See pages 288, 289, and 291 of the textbook for examples of the expected format.
   - Briefly analyze the time complexity of your solution to verify that it runs in polynomial time.

   (4 points)

(b) Consider a version of this problem called *PLANETSP* where the "cities" are points that lie on a two-dimensional plane. Let $S = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$ be this set of points, where each $x_i, y_i \in \mathbb{N}$. The decision problem is

$$\begin{aligned} PLANETSP = \{\langle S, D\rangle \mid\ & S = \{(x_1, y_1), \ldots, (x_k, y_k)\} \text{ and there is a tour of } S \\ & \text{whose total length is no more than } D\}. \end{aligned}$$

Prove that $PLANETSP \in \text{NP}$ using both of the following techniques.

- Give a polynomial time verifier that verifies it. Briefly justify how you know the length of an accepted certificate is polynomial in the length of $\langle S, D\rangle$, and analyze the time complexity of your solution to verify that it runs in polynomial time.
- Give a nondeterministic polynomial time algorithm that decides it. Briefly analyze the time complexity of your solution to verify that it runs in polynomial time.

See pages 295, 296, and 297 of the textbook for examples of the expected format.

You may assume that there is a function d that is computable in polynomial time, where $d((x_i, y_i), (x_j, y_j))$ is an acceptable approximation of the distance between points (x_i, y_i) and (x_j, y_j).

(4 points)

(c) Consider the problem $\overline{PLANETSP}$, which is a member of coNP.

$$\overline{PLANETSP} = \{\langle S, D\rangle \mid S = \{(x_1, y_1), \ldots, (x_k, y_k)\} \text{ and there is \underline{not} a tour of } S \text{ whose total length is no more than } D\} \cup \{w \mid w \text{ is not a valid encoding of any } S, D\}$$

Show that this problem is in EXPTIME by giving an exponential-time algorithm that decides it. **(4 points)**

2. Consider a version of the traveling salesman problem where the "cities" are vertices of an undirected weighted graph G, where $w(u, v) \geq 0$ is the distance between cities u and v. Note that this "distance" does not necessarily represent a Euclidean distance between cities, and the vertices do not necessarily correspond to points on a plane, as in the previous problems.

$$\mathit{TSP} = \{\langle G, w, D\rangle \mid G \text{ is an undirected weighted graph with weights } w \text{ and there is a tour through } G \text{ of length at most } D\}$$

Assume that w is some data structure that defines the weights of the undirected graph G with the property that $w(u, v) = w(v, u)$. If edge $\{u, v\}$ is not in G, the value stored in $w(u, v)$ is ignored and is treated as $w(u, v) = \infty$.

Prove that this problem is NP-complete by

- showing that it is a member of NP, and
- describing a polynomial time reduction from *UHAMPATH*.

(6 points)

3. In the *knapsack problem*, you are given a knapsack with maximum weight capacity W kilograms, and a set of k items,

$$S = \{(w_1, v_1), \ldots, (w_k, v_k)\}$$

where w_i is the weight (in kilograms) of item i and v_i is the value (in dollars) of item i. The decision version of this problem is: Is there a subset of the items with total weight at most W and total value at least V? More formally,

$$KNAPSACK = \left\{ \langle S, W, V \rangle \mid \exists T \subseteq S \sum_{(w,v) \in T} w \leq W \text{ and } \sum_{(w,v) \in T} v \geq V \right\}.$$

Prove that this problem is NP-complete by

- showing that it is a member of NP, and
- describing a polynomial time reduction from any of the NP-complete problems presented in Chapter 7.

(6 points)

Changelog

- **Apr 20:** Added clarifications to 2.